Subject Name: **Strength of materials**

Subject Code: **CE-3003**

Semester: **3rd**

Unit-I

Simple Stress and Strains: Concept of Elastic body stress and Strain, Hooke's law, various types of stress and strains, Elastic constants, Stresses in compound bars, composite and tapering bars, Temperature stresses. Complex Stress and Strains- Two dimensional and three dimensional stress system. Normal and tangential stresses, Principal Planes, Principal Stresses and Strains, Mohr's circle of stresses

Engineering science is usually subdivided into number of topics such as

1. Solid Mechanics
2. Fluid Mechanics
3. Heat Transfer
4. Properties of materials and soon Although there are close links between them in terms of the physical principles involved and methods of analysis employed.

The solid mechanics as a subject may be defined as a branch of applied mechanics that deals with behaviors of solid bodies subjected to various types of loadings. This is usually subdivided into further two streams i.e. Mechanics of rigid bodies or simply Mechanics and Mechanics of deformable solids.

The mechanics of deformable solids which is branch of applied mechanics is known by several names i.e. strength of materials, mechanics of materials etc.

Mechanics of rigid bodies:

The mechanics of rigid bodies is primarily concerned with the static and dynamic behavior under external forces of engineering components and systems which are treated as infinitely strong and undeformable Primarily we deal here with the forces and motions associated with particles and rigid bodies.

Mechanics of deformable solids :

Mechanics of solids:

The mechanics of deformable solids is more concerned with the internal forces and associated changes in the geometry of the components involved. Of particular importance are the properties of the materials used, the strength of which will determine whether the components fail by breaking in service, and the stiffness of which will determine whether the amount of deformation they suffer is acceptable. Therefore, the subject of mechanics of materials or strength of materials is central to the whole activity of engineering design. Usually the objectives in analysis here will be the determination of the stresses, strains, and deflections produced by loads. Theoretical analyses and experimental results have equal roles in this field.

Analysis of stress and strain:

Concept of stress: Let us introduce the concept of stress as we know that the main problem of engineering mechanics of material is the investigation of the internal resistance of the body, i.e. the nature of forces set up within a body to balance the effect of the externally applied forces.

The externally applied forces are termed as loads. These externally applied forces may be due to any one of the reason.

(i) Due to service conditions

(ii) Due to environment in which the component works

(iii) Through contact with other members

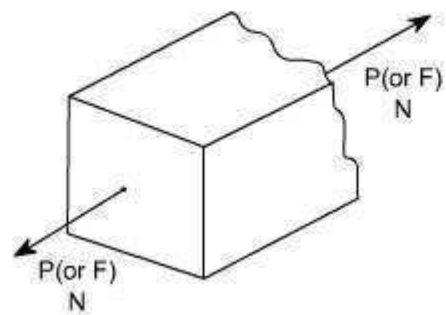
(iv) Due to fluid pressures

(v) Due to gravity or inertia forces.

As we know that in mechanics of deformable solids, externally applied forces acts on a body and body suffers a deformation. From equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particles of material due to cohesion.

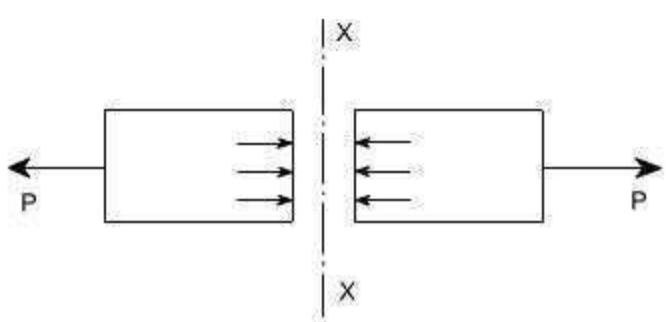
These internal forces give rise to a concept of stress. Therefore, let us define a stress Therefore; let us define a term stress

Stress:



Let us consider a rectangular bar of some cross – sectional area and subjected to some load or force (in Newtons)

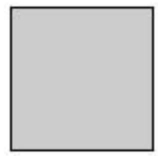
Let us imagine that the same rectangular bar is assumed to be cut into two halves at section XX. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been shown



Now stress is defined as the force intensity or force per unit area. Here we use a symbol s to represent the stress.

$$\sigma = \frac{P}{A}$$

Where A is the area of the X – section



Here we are using an assumption that the total force or total load carried by the rectangular bar is uniformly distributed over its cross – section.

But the stress distributions may be for from uniform, with local regions of high stress known as stress concentrations.

If the force carried by a component is not uniformly distributed over its cross – sectional area, A, we must consider a small area, 'dA' which carries a small load dP, of the total force 'P', Then definition of stress is

$$\boxed{\sigma = \frac{\delta F}{\delta A}}$$

As a particular stress generally holds true only at a point, therefore it is defined mathematically as

$$\boxed{\sigma = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}}$$

Units:

The basic units of stress in S.I units i.e. (International system) are N / m^2 (or Pa)

MPa = 10^6 Pa

GPa = 10^9 Pa

KPa = 10^3 Pa

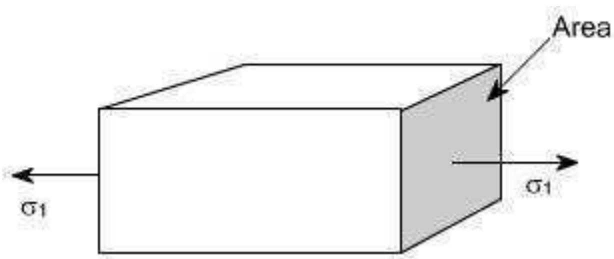
Some times N / mm^2 units are also used, because this is an equivalent to MPa. While US customary unit is pound per square inch psi.

<u>TYPES OF STRESSES :</u>

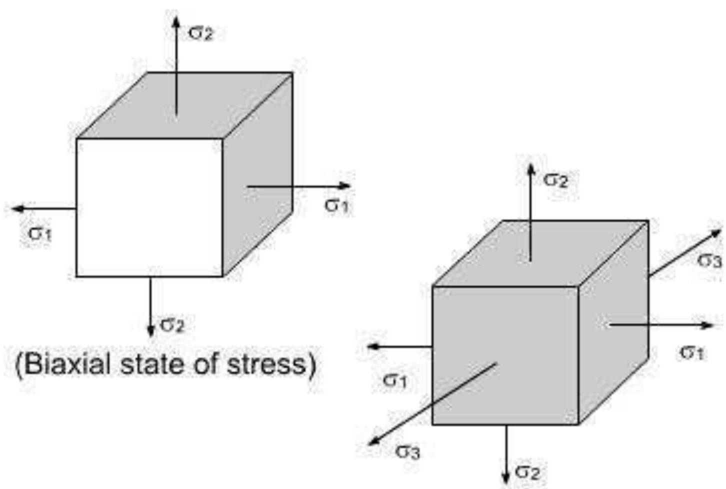
only two basic stresses exists : (1) normal stress and (2) shear shear stress. Other stresses either are similar to these basic stresses or are a combination of this e.g. bending stress is a combination tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress.

Let us define the normal stresses and shear stresses in the following sections.

Normal stresses: We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses. The normal stresses are generally denoted by a Greek letter (s)



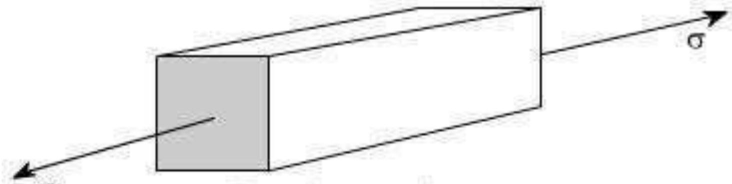
This is also known as uniaxial state of stress, because the stresses acts only in one direction however, such a state rarely exists, therefore we have biaxial and triaxial state of stresses where either the two mutually perpendicular normal stresses acts or three mutually perpendicular normal stresses acts as shown in the figures below :



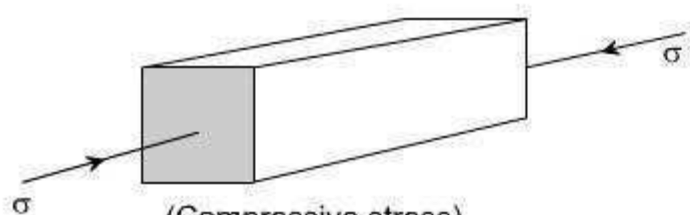
(Biaxial state of stress)
(Triaxial state of stress)

Tensile or compressive stresses:

The normal stresses can be either tensile or compressive whether the stresses acts out of the area or into the area

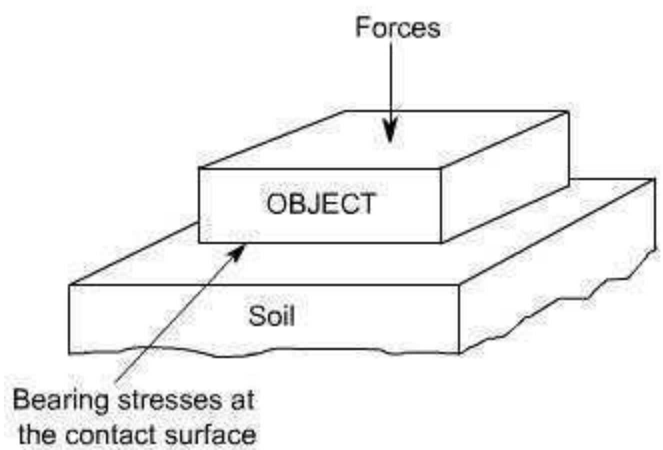


(Tensile stress)



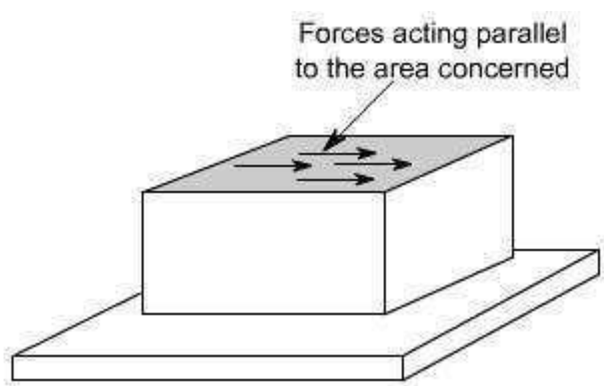
(Compressive stress)

Bearing Stress: When one object presses against another, it is referred to a bearing stress (They are in fact the compressive stresses).



Shear stresses:

Let us consider now the situation, where the cross – sectional area of a block of material is subject to a distribution of forces which are parallel, rather than normal, to the area concerned. Such forces are associated with a shearing of the material, and are referred to as shear forces. The resulting force interistes are known as shear stresses.



The resulting force intensities are known as shear stresses, the mean shear stress being equal to

$$\tau = \frac{P}{A}$$

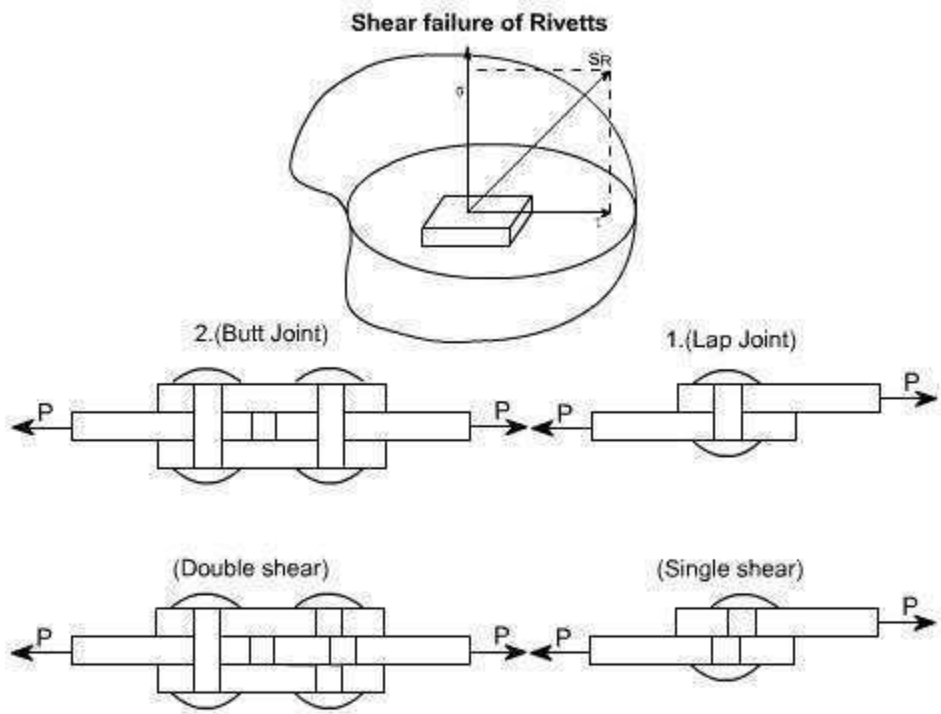
Where P is the total force and A the area over which it acts.

As we know that the particular stress generally holds good only at a point therefore we can define shear stress at a point as

$$\tau = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

The Greek symbol t (tau) (suggesting tangential) is used to denote shear stress.

However, it must be borne in mind that the stress (resultant stress) at any point in a body is basically resolved into two components s and t one acts perpendicular and other parallel to the area concerned, as it is clearly defined in the following figure.

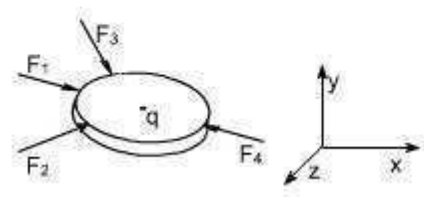


The single shear takes place on the single plane and the shear area is the cross - sectional of the rivet, whereas the double shear takes place in the case of Butt joints of rivets and the shear area is the twice of the X - sectional area of the rivet.

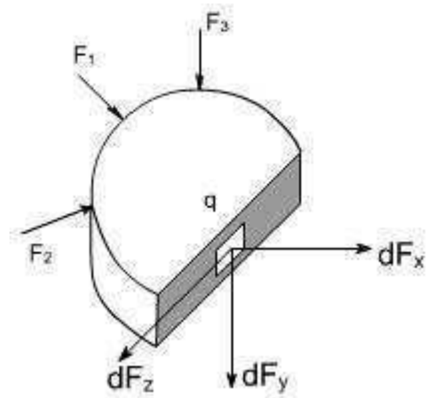
ANALYSIS OF STERSSES

General State of stress at a point:

Stress at a point in a material body has been defined as a force per unit area. But this definition is some what ambiguous since it depends upon what area we consider at that point. Let us, consider a point 'q' in the interior of the body



Let us pass a cutting plane through a pont 'q' perpendicular to the x - axis as shown below



The corresponding force components can be shown like this

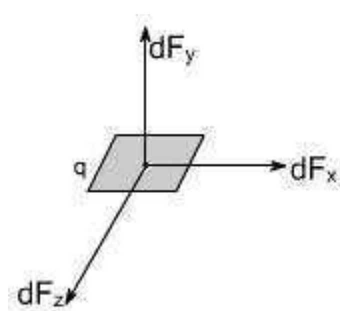
$dF_x = s_{xx}. da_x$

$dF_y = t_{xy}. da_x$

$dF_z = t_{xz}. da_x$

where da_x is the area surrounding the point 'q' when the cutting plane ^ r is to x - axis.

In a similar way it can be assumed that the cutting plane is passed through the point 'q' perpendicular to the y - axis. The corresponding force components are shown below



The corresponding force components may be written as

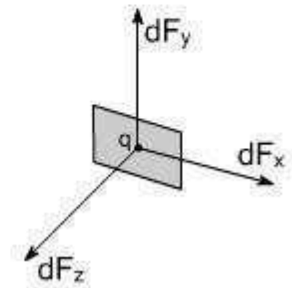
$dF_x = t_{yx}. da_y$

$dF_y = s_{yy}. da_y$

$dF_z = t_{yz}. da_y$

where da_y is the area surrounding the point 'q' when the cutting plane ^ ʳ is to y - axis.

In the last it can be considered that the cutting plane is passed through the point 'q' perpendicular to the z - axis.



The corresponding force components may be written as

$dF_x = t_{zx}. da_z$

$dF_y = t_{zy}. da_z$

$dF_z = s_{zz}. da_z$

where da_z is the area surrounding the point 'q' when the cutting plane ^ ʳ is to z - axis.

Thus, from the foregoing discussion it is amply clear that there is nothing like stress at a point 'q' rather we have a situation where it is a combination of state of stress at a point q. Thus, it becomes imperative to understand the term state of stress at a point 'q'. Therefore, it becomes easy to express state of stress by the scheme as discussed earlier, where the stresses on the three mutually perpendicular planes are labeled in the manner as shown earlier. the state of stress as depicted earlier is called the general or a triaxial state of stress that can exist at any interior point of a loaded body.

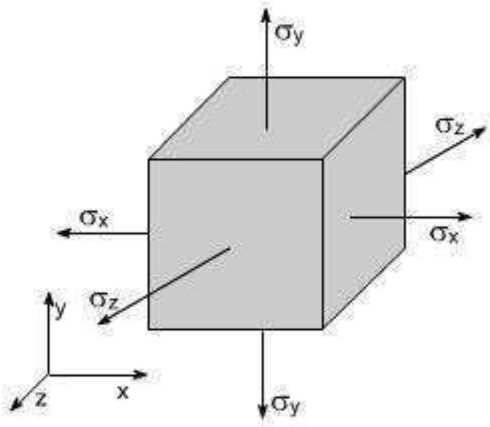
Before defining the general state of stress at a point. Let us make over selves conversant with the notations for the stresses.

We have already chosen to distinguish between normal and shear stress with the help of symbols s and t .

Cartesian - co-ordinate system

In the Cartesian co-ordinates system, we make use of the axes, X, Y and Z

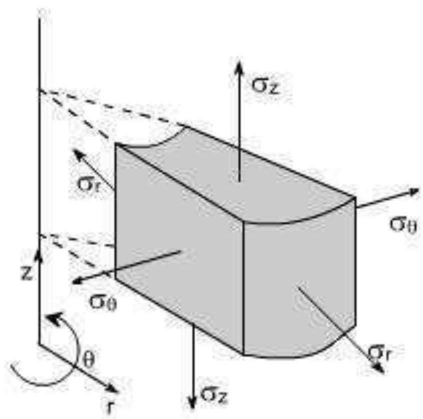
Let us consider the small element of the material and show the various normal stresses acting the faces



Thus, in the Cartesian co-ordinates system the normal stresses have been represented by s_x, s_y and s_z.

Cylindrical - co-ordinate system

In the Cylindrical - co-ordinate system we make use of co-ordinates r, q and Z.



Thus, in the Cylindrical co-ordinates system, the normal stresses i.e components acting over a element is being denoted by s_r, $s_q$and s_z.

Sign convention : The tensile forces are termed as (+ve) while the compressive forces are termed as negative (-ve).

First sub – script : it indicates the direction of the normal to the surface.

Second subscript : it indicates the direction of the stress.

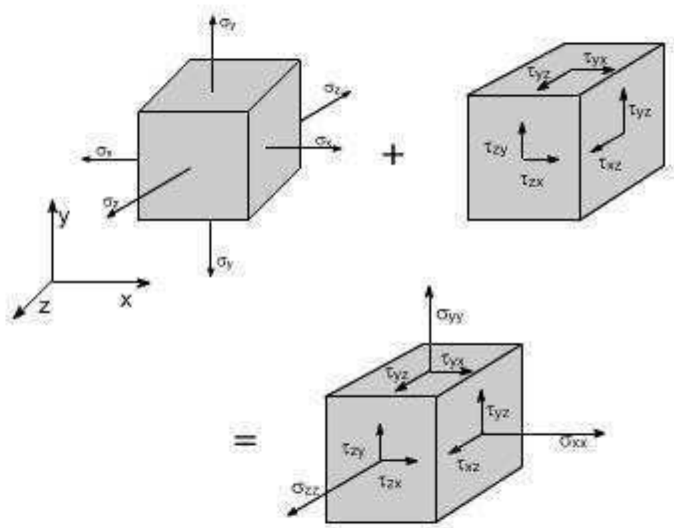
It may be noted that in the case of normal stresses the double script notation may be dispensed with as the direction of the normal stress and the direction of normal to the surface of the element on which it acts is the same. Therefore, a single subscript notation as used is sufficient to define the normal stresses.

Shear Stresses : With shear stress components, the single subscript notation is not practical, because such stresses are in direction parallel to the surfaces on which they act. We therefore have two directions to specify, that of normal to the surface and the stress itself. To do this, we stress itself. To do this, we attach two subscripts to the symbol ' t' , for shear stresses.

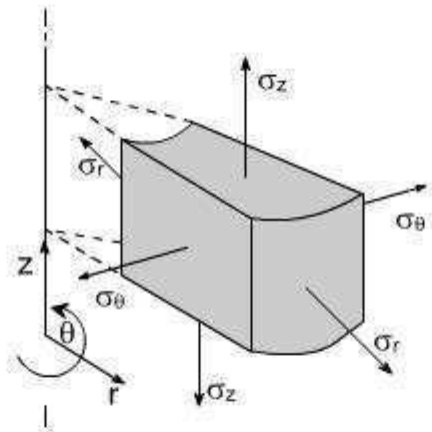
In cartesian and polar co-ordinates, we have the stress components as shown in the figures.

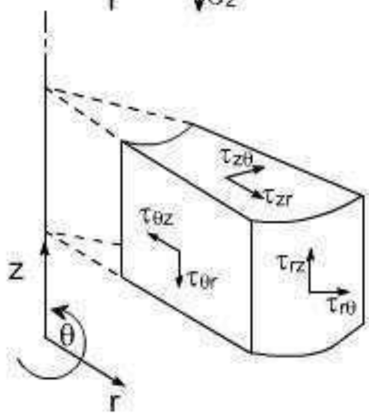
t_{xy} , t_{yx} , t_{yz} , t_{zy} , t_{zx} , t_{xz}

t_{rq} , t_{qr} , t_{qz} , t_{zq} ,t_{zr} , t_{rz}

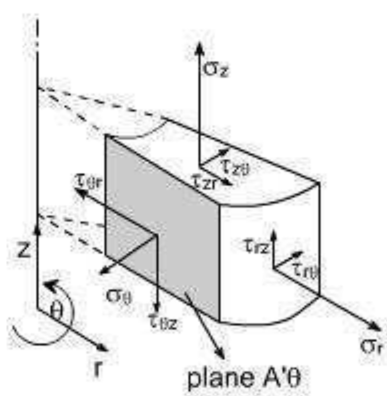


So as shown above, the normal stresses and shear stress components indicated on a small element of material seperately has been combined and depicted on a single element. Similarly for a cylindrical co-ordinate system let us shown the normal and shear stresses components separately.





Now let us combine the normal and shear stress components as shown below :



Now let us define the state of stress at a point formally.

State of stress at a point :

By state of stress at a point, we mean information which is required at that point such that it remains under equilibrium. or simply a general state of stress at a point involves all the normal stress components, together with all the shear stress components as shown in earlier figures.

Therefore, we need nine components, to define the state of stress at a point

s_x t_{xy} t_{xz}

s_y t_{yx} t_{yz}

s_z t_{zx} t_{zy}

If we apply the conditions of equilibrium which are as follows:

å $F_x = 0$; å $M_x = 0$

å $F_y = 0$; å $M_y = 0$

å $F_z = 0$; å $M_z = 0$

Then we get

$t_{xy} = t_{yx}$

$t_{yz} = t_{zy}$

$t_{zx} = t_{xy}$

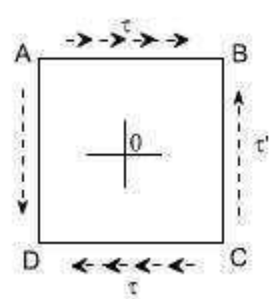
Then we will need only six components to specify the state of stress at a point i.e

s_x , s_y, s_z , t_{xy} , t_{yz} , t_{zx}

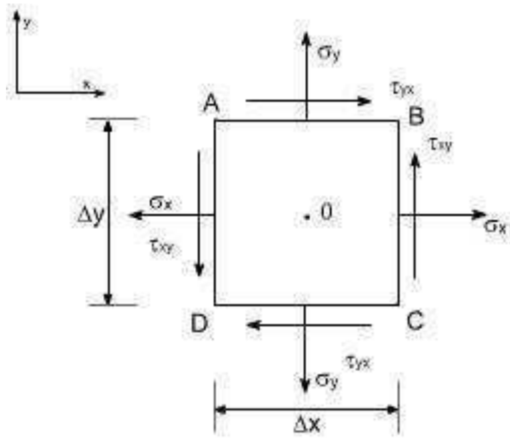
Now let us define the concept of complementary shear stresses.

Complementary shear stresses:

The existence of shear stresses on any two sides of the element induces complementary shear stresses on the other two sides of the element to maintain equilibrium.



on planes AB and CD, the shear stress t acts. To maintain the static equilibrium of this element, on planes AD and BC, t' should act, we shall see that t' which is known as the complementary shear stress would come out to equal and opposite to the t . Let us prove this thing for a general case as discussed below:



The figure shows a small rectangular element with sides of length Dx, Dy parallel to x and y directions. Its thickness normal to the plane of paper is Dz in z – direction. All nine normal and shear stress components may act on the element, only those in x and y directions are shown.

Sign convections for shear stresses:

Direct stresses or normal stresses

- Tensile +ve

- Compressive –ve

Shear stresses:

- Tending to turn the element C.W +ve.

- Tending to turn the element C.C.W – ve.

The resulting forces applied to the element are in equilibrium in x and y direction. (Although other normal and shear stress components are not shown, their presence does not affect the final conclusion).

Assumption : The weight of the element is neglected.

Since the element is a static piece of solid body, the moments applied to it must also be in equilibrium. Let 'O' be the centre of the element. Let us consider the axis through the point 'O'. the resultant force associated with normal stresses s_x and s_y acting on the sides of the element each pass through this axis, and therefore, have no moment.

Now forces on top and bottom surfaces produce a couple which must be balanced by the forces on left and right hand faces

Thus,

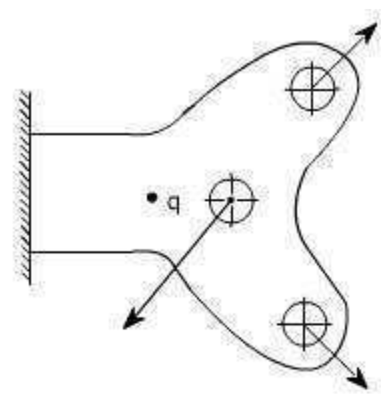
t_{yx} . D x . D z . D y = t_{xy} . D x . D z . D y

$$\boxed{\tau_{yx} = \tau_{xy}}$$

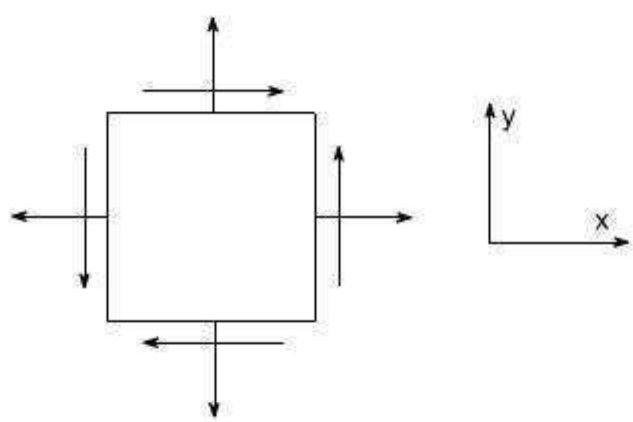
In other word, the complementary shear stresses are equal in magnitude. The same form of relationship can be obtained for the other two pair of shear stress components to arrive at the relations

$$\boxed{\begin{array}{l}\tau_{zy} = \tau_{zy} \\ \tau_{zx} = \tau_{xz}\end{array}}$$

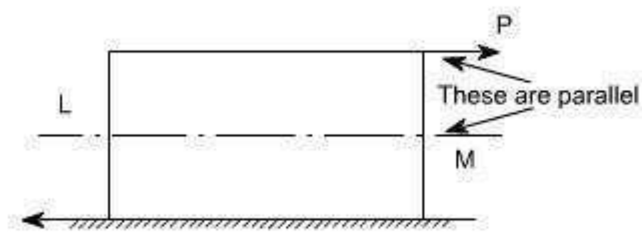
Analysis of Stresses:



Consider a point 'q' in some sort of structural member like as shown in figure below. Assuming that at point exist. 'q' a plane state of stress exist. i.e. the state of state stress is to describe by a parameters s_x, s_y and t_{xy} These stresses could be indicate a on the two dimensional diagram as shown below:



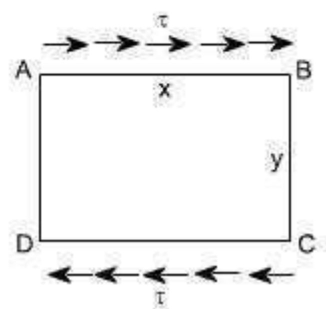
This is a commen way of representing the stresses. It must be realize a that the material is unaware of what we have called the x and y axes. i.e. the material has to resist the loads irrespective less of how we wish to name them or whether they are horizontal, vertical or otherwise further more, the material will fail when the stresses exceed beyond a permissible value. Thus, a fundamental problem in engineering design is to determine the maximum normal stress or maximum shear stress at any particular point in a body. There is no reason to believe apriori that s_x, s_y and t_{xy} are the maximum value. Rather the maximum stresses may associates themselves with some other planes located at 'q'. Thus, it becomes imperative to determine the values of s_q and t_q. In order tto achieve this let us consider the following.



Shear stress:

If the applied load P consists of two equal and opposite parallel forces not in the same line, than there is a tendency for one part of the body to slide over or shear from the other part across any section LM. If the cross section at LM measured parallel to the load is A, then the average value of shear stress t = P/A . The shear stress is tangential to the area over which it acts.

If the shear stress varies then at a point then t may be defined as $\tau = \lim_{\delta A \to 0} \frac{\delta P}{\delta A}$



Complementary shear stress:

Let ABCD be a small rectangular element of sides x, y and z perpendicular to the plane of paper let there beshear stress acting on planes AB and CD

It is obvious that these stresses will from a couple (t . xz)y which can only be balanced by tangential forces on planes AD and BC. These are known as complementary shear stresses. i.e. the existence of shear stresses on sides AB and CD of the element implies that there must also be complementary shear stresses on to maintain equilibrium.

Let t' be the complementary shear stress induced on planes AD and BC. Then for the equilibrium (t . xz)y = t' (yz)x

$$\boxed{t = t'}$$

Thus, every shear stress is accompanied by an equal complementary shear stress.

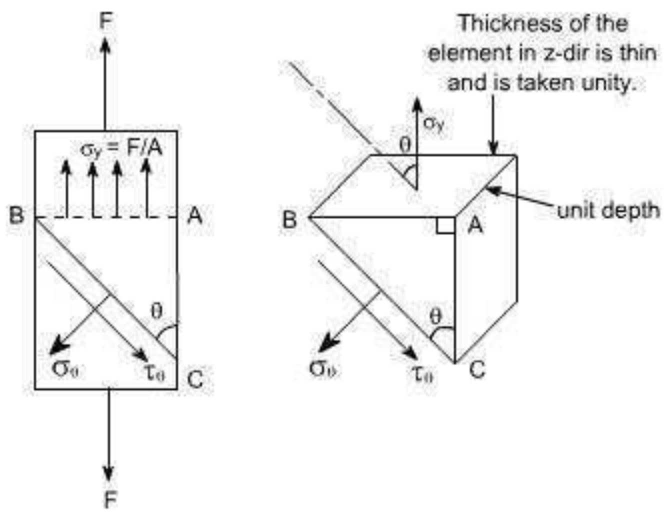
Stresses on oblique plane: Till now we have dealt with either pure normal direct stress or pure shear stress. In many instances, however both direct and shear stresses acts and the resultant stress across any section will be neither normal nor tangential to the plane.

A plane stse of stress is a 2 dimensional stae of stress in a sense that the stress components in one direction are all zero i.e

$$s_z = t_{yz} = t_{zx} = 0$$

examples of plane state of stress includes plates and shells.

Consider the general case of a bar under direct load F giving rise to a stress s_y vertically



The stress acting at a point is represented by the stresses acting on the faces of the element enclosing the point.

The stresses change with the inclination of the planes passing through that point i.e. the stress on the faces of the element vary as the angular position of the element changes.

Let the block be of unit depth now considering the equilibrium of forces on the triangle portion ABC

Resolving forces perpendicular to BC, gives

$$s_q.BC.1 = s_y\sin q \;.\; AB \;.\; 1$$

but

$$AB/BC = \sin q \text{ or } AB = BC\sin q$$

Substituting this value in the above equation, we get

$$s_q.BC.1 = s_y\sin q \;.\; BC\sin q \;.\; 1 \text{ or } \boxed{\sigma_\theta = \sigma_y\ .\sin^2 2\theta} \qquad (1)$$

Now resolving the forces parallel to BC

$$t_q.BC.1 = s_y \; cosq \; . \; ABsinq \; . \; 1$$

$$\text{again } AB = BCcosq$$

$$t_q.BC.1 = s_y cosq \; . \; BCsinq \; . \; 1 \text{ or } t_q = s_y sinqcosq$$

$$\boxed{\tau_\theta = \frac{1}{2}.\sigma_y.\sin 2\theta} \quad (2)$$

If q = 90^0 the BC will be parallel to AB and t_q = 0, i.e. there will be only direct stress or normal stress.

By examining the equations (1) and (2), the following conclusions may be drawn

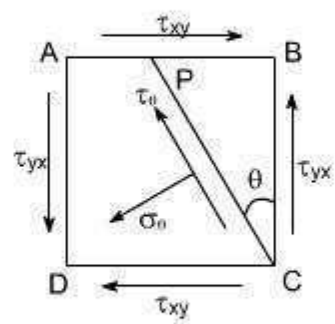
(i) The value of direct stress s_q is maximum and is equal to s_y when q = 90^0.

(ii) The shear stress t_q has a maximum value of 0.5 s_y when q = 45^0

(iii) The stresses s_q and s_q are not simply the resolution of s_y

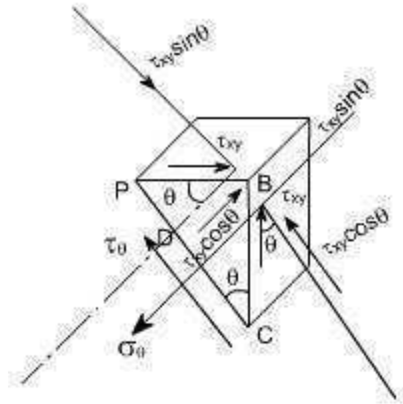
Material subjected to pure shear:

Consider the element shown to which shear stresses have been applied to the sides AB and DC



Complementary shear stresses of equal value but of opposite effect are then set up on the sides AD and BC in order to prevent the rotation of the element. Since the applied and complementary shear stresses are of equal value on the x and y planes. Therefore, they are both represented by the symbol t_{xy}.

Now consider the equilibrium of portion of PBC



Assuming unit depth and resolving normal to PC or in the direction of s_q

$$s_q.PC.1 = t_{xy}.PB.cosq.1+ t_{xy}.BC.sinq.1$$

$$= t_{xy}.PB.cosq + t_{xy}.BC.sinq$$

Now writing PB and BC in terms of PC so that it cancels out from the two sides

$$PB/PC = sinq \quad BC/PC = cosq$$

$s_q.PC.1 = t_{xy}.cosqsinqPC+ t_{xy}.cosq.sinqPC$

$$s_q = 2t_{xy}sinqcosq$$

$$s_q = t_{xy}.2.sinqcosq$$

$$\boxed{\sigma_\theta = \tau_{xy}.\sin 2\theta} \qquad (1)$$

Now resolving forces parallel to PC or in the direction t_q.then

$$t_{xy}PC . 1 = t_{xy} . PBsinq - t_{xy} . BCcosq$$

-ve sign has been put because this component is in the same direction as that of t_q.

again converting the various quantities in terms of PC we have

$$t_{xy}PC . 1 = t_{xy} . PB.sin^2q - t_{xy} . PCcos^2q$$

$$= -[t_{xy} (cos^2q - sin^2q)]$$

$$= -t_{xy}cos2q \text{ or } \boxed{\tau_\theta = -\tau_{xy}\cos 2\theta} \qquad (2)$$

the negative sign means that the sense of t_q is opposite to that of assumed one. Let us examine the equations (1) and (2) respectively

From equation (1) i.e,

$$s_q = t_{xy} \sin 2q$$

The equation (1) represents that the maximum value of s_q is t_{xy} when $q = 45^0$.

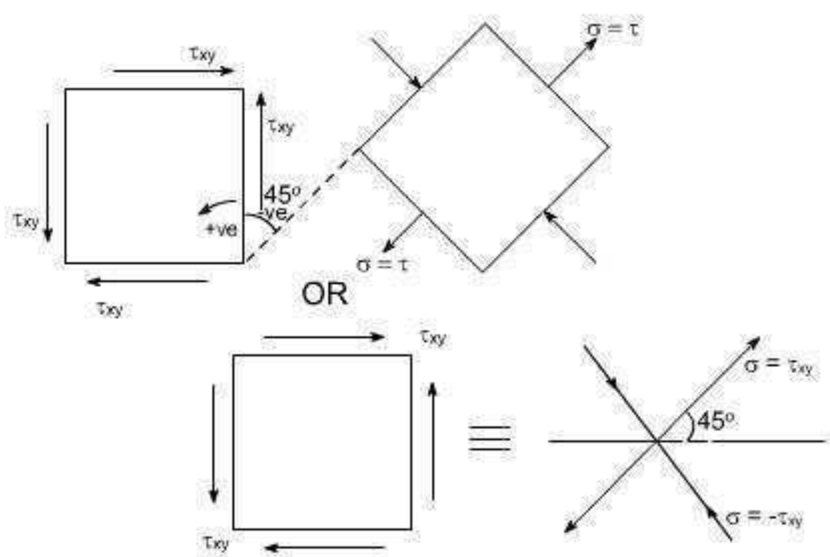
Let us take into consideration the equation (2) which states that

$$t_q = - t_{xy} \cos 2q$$

It indicates that the maximum value of t_q is t_{xy} when $q = 0^0$ or 90^0. it has a value zero when $q = 45^0$.

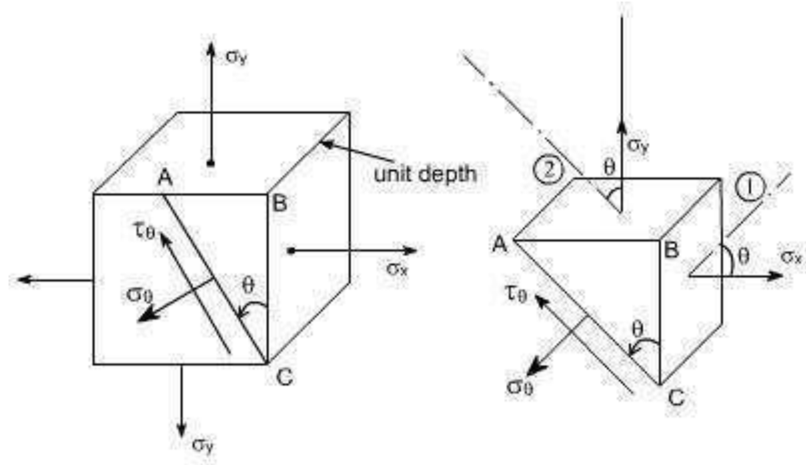
From equation (1) it may be noticed that the normal component s_q has maximum and minimum values of $+t_{xy}$(tension) and $-t_{xy}$ (compression) on plane at $\pm 45^0$ to the applied shear and on these planes the tangential component t_q is zero.

Hence the system of pure shear stresses produces and equivalent direct stress system, one set compressive and one tensile each located at 45^0 to the original shear directions as depicted in the figure below:



Material subjected to two mutually perpendicular direct stresses:

Now consider a rectangular element of unit depth, subjected to a system of two direct stresses both tensile, s_x and $s_y$acting right angles to each other.



for equilibrium of the portion ABC, resolving perpendicular to AC

$$s_q \text{ . AC.1} = s_y \sin q \text{ . AB.1} + s_x \cos q \text{ . BC.1}$$

converting AB and BC in terms of AC so that AC cancels out from the sides

$$s_q = s_y \sin^2 q + s_x \cos^2 q$$

Further, recalling that $\cos^2 q$ - $\sin^2 q$ = cos2q or (1 - cos2q)/2 = $\sin^2 q$

$$\text{Similarly } (1 + \cos 2q)/2 = \cos^2 q$$

Hence by these transformations the expression for s_q reduces to

$$= 1/2 s_y (1 - \cos 2q) + 1/2 s_x (1 + \cos 2q)$$

On rearranging the various terms we get

$$\boxed{\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta} \qquad (3)$$

Now resolving parallel to AC

$$s_q\text{.AC.1} = -t_{xy}\text{..cos}q\text{.AB.1} + t_{xy}\text{.BC.sin}q\text{.1}$$

The – ve sign appears because this component is in the same direction as that of AC.

Again converting the various quantities in terms of AC so that the AC cancels out from the two sides.

$$\tau_\theta.AC.1 = [\tau_x \cos\theta \sin\theta - \sigma_y \sin\theta \cos\theta]AC$$

$$\tau_\theta = (\sigma_x - \sigma_y)\sin\theta\cos\theta$$

$$= \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

$$\text{or} \quad \boxed{\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta} \qquad (4)$$

Conclusions :

The following conclusions may be drawn from equation (3) and (4)

(i) The maximum direct stress would be equal to s_x or s_y which ever is the greater, when q = 0^0 or 90^0

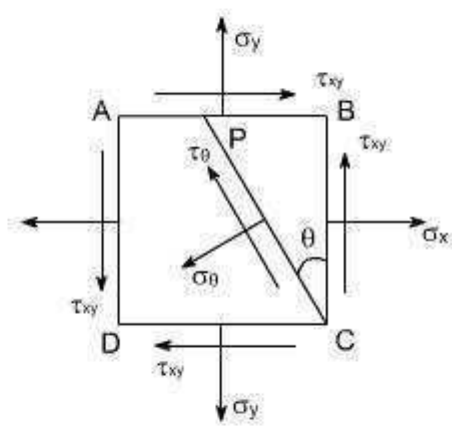
(ii) The maximum shear stress in the plane of the applied stresses occurs when q = 45^0

$$\tau_{max} = \frac{(\sigma_x - \sigma_y)}{2}$$

Material subjected to combined direct and shear stresses:

Now consider a complex stress system shown below, acting on an element of material.

The stresses s_x and s_y may be compressive or tensile and may be the result of direct forces or as a result of bending.The shear stresses may be as shown or completely reversed and occur as a result of either shear force or torsion as shown in the figure below:



As per the double subscript notation the shear stress on the face BC should be notified as t_{yx} , however, we have already seen that for a pair of shear stresses there is a set of complementary shear stresses generated such that $t_{yx} = t_{xy}$

By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

(i) Material subjected to pure stae of stress shear. In this case the various formulas deserved are as follows sq = tyx sin2 q

tq = - tyx cos 2 q

(ii) Material subjected to two mutually perpendicular direct stresses. In this case the various formula's derived are as follows.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta$$
$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

To get the required equations for the case under consideration,let us add the respective equations for the above two cases such that

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

These are the equilibrium equations for stresses at a point. They do not depend on material proportions and are equally valid for elastic and inelastic behaviour

This eqn gives two values of 2q that differ by 180^0 .Hence the planes on which maximum and minimum normal stresses occurate 90^0 apart.

For σ_θ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$
$$= 0$$

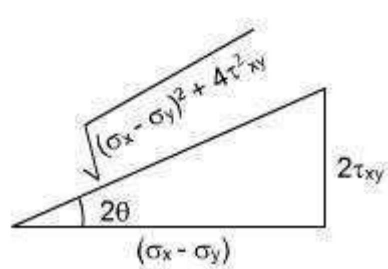
i.e. $-(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$

$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$

Thus, $\boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$

From the triangle it may be determined

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$
$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$



Substituting the values of cos2 q and sin2 q in equation (5) we get

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cdot\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} + \frac{\tau_{xy}\cdot 2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\cdot\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} + \frac{1}{2}\frac{4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

or

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\cdot\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\cdot\frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Hence we get the two values of σ_θ, which are designated σ_1 as σ_2 and respectively,therefore

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

The σ_1 and σ_2 are termed as the principle stresses of the system.

Substituting the values of cos2θ and sin2θ in equation (6) we see that

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}\cdot(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = 0$$

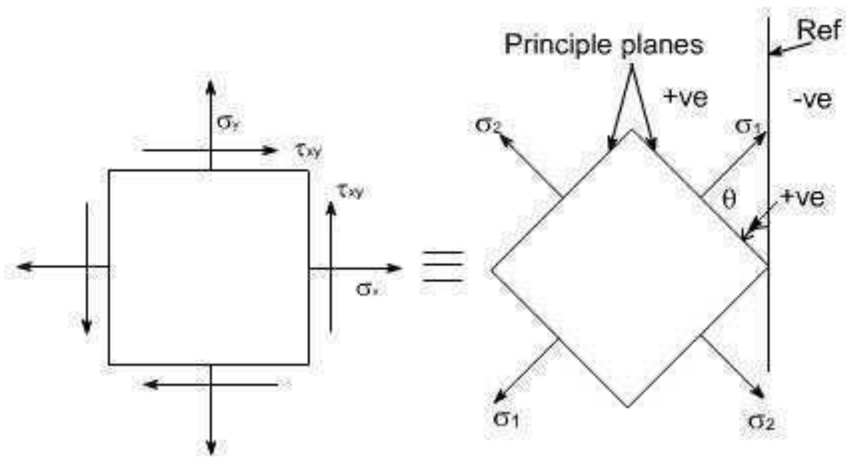
This shows that the values oshear stress is zero on the principal planes.

Hence the maximum and minimum values of normal stresses occur on planes of zero shearing stress. The maximum and minimum normal stresses are called the principal stresses, and the planes on which they act are called principal plane the solution of equation

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

will yield two values of 2q separated by 180^0 i.e. two values of q separated by 90^0 .Thus the two principal stresses occur on mutually perpendicular planes termed principal planes.

Therefore the two – dimensional complex stress system can now be reduced to the equivalent system of principal stresses.



Let us recall that for the case of a material subjected to direct stresses the value of maximum shear stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_x - \sigma_y)$ at $\theta = 45^0$, Thus, for a 2-dimensional state of stress, subjected to principle stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_1 - \sigma_2)$, on substituting the values if σ_1 and σ_2, we get

$$\tau_{max^m} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Alternatively this expression can also be obtained by differentiating the expression for τ_θ with respect to θ i.e.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta.2 + \tau_{xy}\sin 2\theta.2$$

$$= 0$$

$$\text{or } (\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_s = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Recalling that

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Thus,

$$\boxed{\tan 2\theta_p . \tan 2\theta_s = 1}$$

Therefore, it can be concluded that the equation (2) is a negative reciprocal of equation (1) hence the roots for the double angle of equation (2) are 90^0 away from the corresponding angle of equation (1).

This means that the angles that angles that locate the plane of maximum or minimum shearing stresses form angles of 45^0 with the planes of principal stresses.

Futher, by making the triangle we get

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

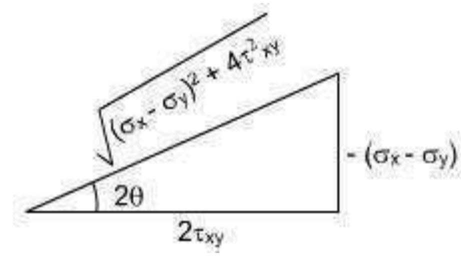
Therefore by substituting the values of $\cos 2\theta$ and $\sin 2\theta$ we have

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}\cdot - \frac{(\sigma_x - \sigma_y).(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$= -\frac{1}{2}\cdot\frac{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$



Because of root the difference in sign convention arises from the point of view of locating the planes on which shear stress act. From physical point of view these sign have no meaning.

The largest stress regard less of sign is always know as maximum shear stress.

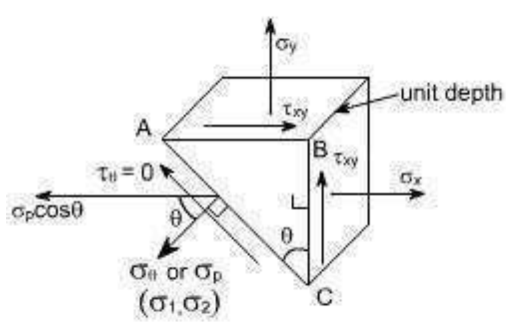
Principal plane inclination in terms of associated principal stress:

We know that the equation $\tan 2\theta_p = \dfrac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$

yields two values of q i.e. the inclination of the two principal planes on which the principal stresses s_1 and s_2 act. It is uncertain,however, which stress acts on which plane unless equation.

$\sigma_\theta = \dfrac{(\sigma_x + \sigma_y)}{2} + \dfrac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$ is used and observing which one of the two principal stresses is obtained.

Alternatively we can also find the answer to this problem in the following manner



Consider once again the equilibrium of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses s_p acts, and the shear stress is zero.

Resolving the forces horizontally we get:

s_x .BC . 1 + t_{xy} .AB . 1 = s_p . cosq . AC   dividing the above equation through by BC we get

$$\sigma_x . + \tau_{xy} \frac{AB}{BC} = \sigma_p . \cos\theta . \frac{AC}{BC}$$

or

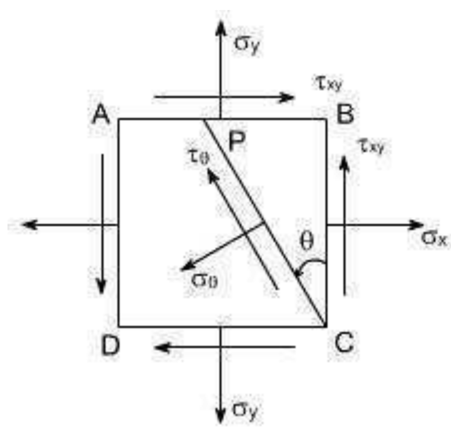
$$\sigma_x + \tau_{xy} \tan\theta = \sigma_p$$

Thus

$$\boxed{\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}}$$

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This grapical representation is very useful in depending the relationships between normal and shear stresses acting on any inclined plane at a point in a stresses body.

To draw a Mohr's stress circle consider a complex stress system as shown in the figure



The above system represents a complete stress system for any condition of applied load in two dimensions

The Mohr's stress circle is used to find out graphically the direct stress s and sheer stress t on any plane inclined at q to the plane on which s_x acts.The direction of q here is taken in anticlockwise direction from the BC.

STEPS:

In order to do achieve the desired objective we proceed in the following manner

(i) Label the Block ABCD.

(ii) Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)

(iii) Plot the stresses on two adjacent faces e.g. AB and BC, using the following sign convention.

Direct stresses - tensile positive; compressive, negative

Shear stresses – tending to turn block clockwise, positive

– tending to turn block counter clockwise, negative

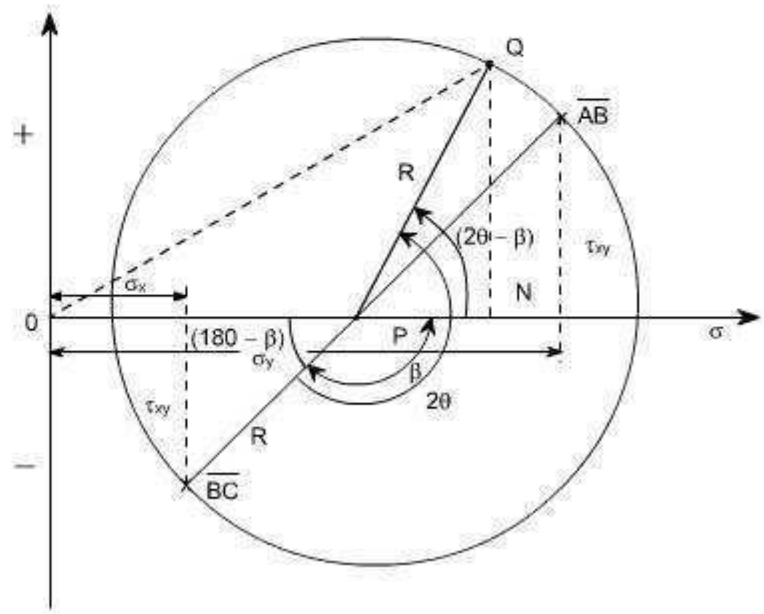
[i.e shearing stresses are +ve when its movement about the centre of the element is clockwise]

This gives two points on the graph which may than be labeled as $\overline{AB}$ and $\overline{BC}$ respectively to denote stresses on these planes.

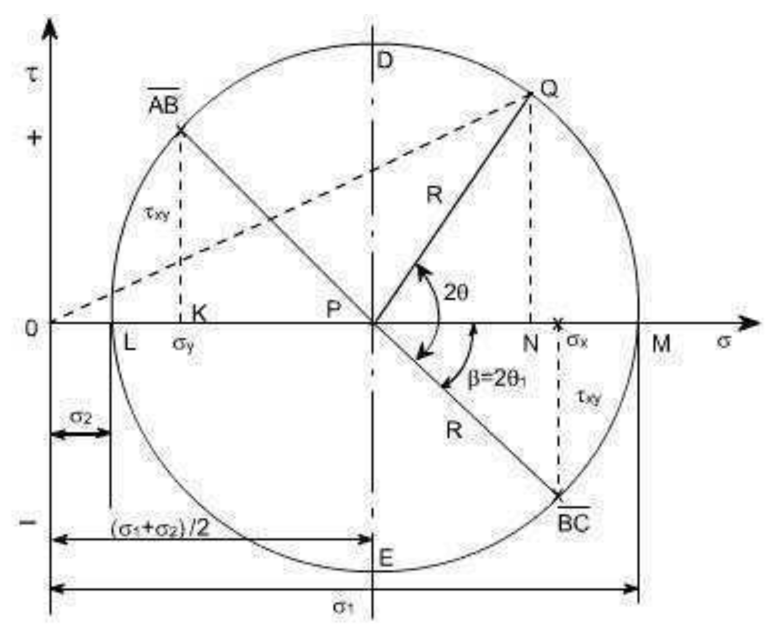
(iv) Join $\overline{AB}$ and $\overline{BC}$.

(v) The point P where this line cuts the s axis is than the centre of Mohr's stress circle and the line joining $\overline{AB}$ and $\overline{BC}$ is diameter. Therefore the circle can now be drawn.

Now every point on the circle then represents a state of stress on some plane through C.



Proof:



Consider any point Q on the circumference of the circle, such that PQ makes an angle 2q with BC, and drop a perpendicular from Q to meet the s axis at N.Then OQ represents the resultant stress on the plane an angle q to BC. Here we have assumed that $s_x > s_y$

Now let us find out the coordinates of point Q. These are ON and QN.

From the figure drawn earlier

$$ON = OP + PN$$

$$OP = OK + KP$$

$$OP = s_y + 1/2 (s_x - s_y)$$

$$= s_y / 2 + s_y / 2 + s_x / 2 + s_y / 2$$

$$= (s_x + s_y) / 2$$

PN = Rcos(2q - b)

hence ON = OP + PN

$= (s_x + s_y) / 2 + \text{Rcos}(2q - b)$

$= (s_x + s_y) / 2 + \text{Rcos2q cosb} + \text{Rsin2qsinb}$

now make the substitutions for Rcosb and Rsinb.

$$R\cos\beta = \frac{(\sigma_x - \sigma_y)}{2}; \; R\sin\beta = \tau_{xy}$$

Thus,

ON = $1/2 (s_x + s_y) + 1/2 (s_x - s_y)\cos 2q + t_{xy}\sin 2q$ (1)

Similarly QM = Rsin(2q - b)

= Rsin2qcosb - Rcos2qsinb

Thus, substituting the values of R cosb and Rsinb, we get

QM = $1/2 (s_x - s_y)\sin 2q - t_{xy}\cos 2q$ (2)

If we examine the equation (1) and (2), we see that this is the same equation which we have already derived analytically

Thus the co-ordinates of Q are the normal and shear stresses on the plane inclined at q to BC in the original stress system.

N.B: Since angle $\overline{BC}$ PQ is 2q on Mohr's circle and not q it becomes obvious that angles are doubled on Mohr's circle. This is the only difference, however, as They are measured in the same direction and from the same plane in both figures.

Further points to be noted are :

(1) The direct stress is maximum when Q is at M and at this point obviously the sheer stress is zero, hence by definition OM is the length representing the maximum principal stresses s_1 and $2q_1$ gives the angle of the plane q_1 from BC. Similar OL is the other principal stress and is represented by s_2

(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle.

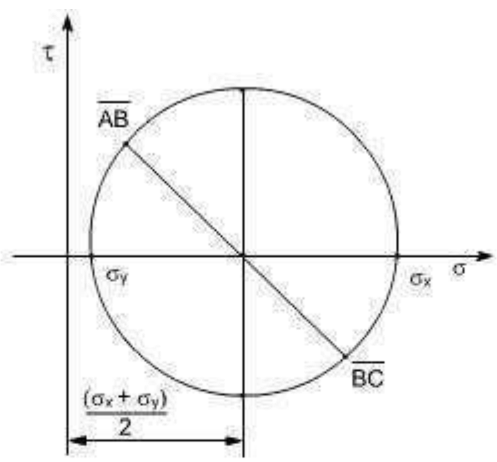
This follows that since shear stresses and complimentary sheer stresses have the same value; therefore the centre of the circle will always lie on the s axis midway between s_x and s_y . [since $+t_{xy}$ & $-t_{xy}$ are shear stress & complimentary shear stress so they are same in magnitude but different in sign.]

(3) From the above point the maximum sheer stress i.e. the Radius of the Mohr's stress circle would be

$$\frac{(\sigma_x - \sigma_y)}{2}$$

While the direct stress on the plane of maximum shear must be mid – may between s_x and s_y i.e

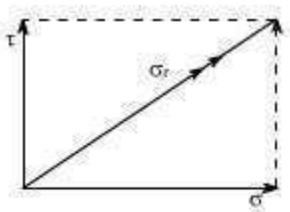
$$\frac{(\sigma_x + \sigma_y)}{2}$$



(4) As already defined the principal planes are the planes on which the shear components are zero.

Therefore are conclude that on principal plane the sheer stress is zero.

(5) Since the resultant of two stress at 90^0 can be found from the parallogram of vectors as shown in the diagram. Thus, the resultant stress on the plane at q to BC is given by OQ on Mohr's Circle.



(6) The graphical method of solution for a complex stress problems using Mohr's circle is a very powerful technique, since all the information relating to any plane within the stressed element is contained in the single construction. It thus, provides a convenient and rapid means of solution. Which is less prone to arithmetical errors and is highly recommended?

ILLUSRATIVE PROBLEMS:

Let us discuss few representative problems dealing with complex state of stress to be solved either analytically or graphically.

PROB 1: A circular bar 40 mm diameter carries an axial tensile load of 105 kN. What is the Value of shear stress on the planes on which the normal stress has a value of 50 MN/m^2 tensile.

Solution:

$$\text{Tensile stress } s_y = F / A = 105 \times 10^3 / p \times (0.02)^2$$

$$= 83.55 \text{ MN/m}^2$$

Now the normal stress on an obliqe plane is given by the relation

$$s_q = s_y \sin^2 q$$

$$50 \times 10^6 = 83.55 \text{ MN/m}^2 \times 10^6 \sin^2 q$$

$$q = 50^0 68'$$

The shear stress on the oblique plane is then given by

$$t_q = 1/2\ s_y \sin 2q$$

$$= 1/2 \times 83.55 \times 10^6 \times \sin 101.36$$

$$= 40.96 \text{ MN/m}^2$$

Therefore the required shear stress is 40.96 MN/m^2

PROB 2:

For a given loading conditions the state of stress in the wall of a cylinder is expressed as follows:

(a) 85 MN/m^2 tensile

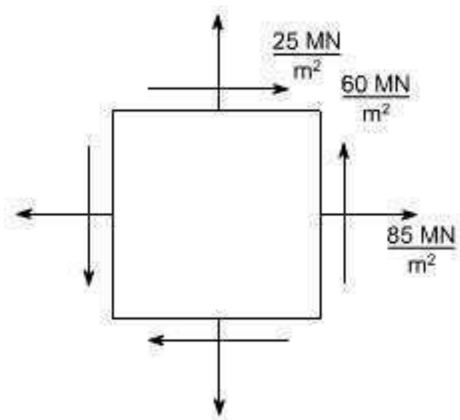
(b) 25 MN/m^2 tensile at right angles to (a)

(c) Shear stresses of 60 MN/m^2 on the planes on which the stresses (a) and (b) act; the sheer couple acting on planes carrying the 25 MN/m^2 stress is clockwise in effect.

Calculate the principal stresses and the planes on which they act. What would be the effect on these results owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged

Solution:

The problem may be attempted both analytically as well as graphically. Let us first obtain the analytical solution



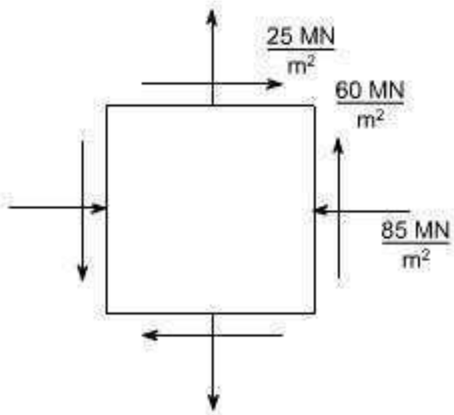
The principle stresses are given by the formula

$$\begin{aligned}
&\sigma_1 \text{ and } \sigma_2 \\
&= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}} \\
&= \frac{1}{2}(85 + 25) \pm \frac{1}{2}\sqrt{(85 + 25)^2 + (4 \times 60^2)} \\
&= 55 \pm \frac{1}{2}.60\sqrt{5} = 55 \pm 67 \\
&\Rightarrow \sigma_1 = 122 \ MN/m^2 \\
&\quad \sigma_2 = -12 \ MN/m^2 (\text{compressive})
\end{aligned}$$

For finding out the planes on which the principle stresses act us the equation $\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$

The solution of this equation will yeild two values q i.e they q_1 and q_2 giving $q_1 = 31^0 71'$ & $q_2 = 121^0 71'$

(b) In this case only the loading (a) is changed i.e. its direction had been changed. While the other stresses remains unchanged hence now the block diagram becomes.



Again the principal stresses would be given by the equation.

$$\sigma_1 \,\&\, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$= \frac{1}{2}(-85 + 25) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= \frac{1}{2}(-60) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)}$$

$$= -30 \pm \frac{1}{2}\sqrt{12100 + 14400}$$

$$= -30 \pm 81.4$$

$$\sigma_1 = 51.4 \text{ MN/m}^2;\ \sigma_2 = -111.4 \text{ MN/m}^2$$

Again for finding out the angles use the following equation.

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

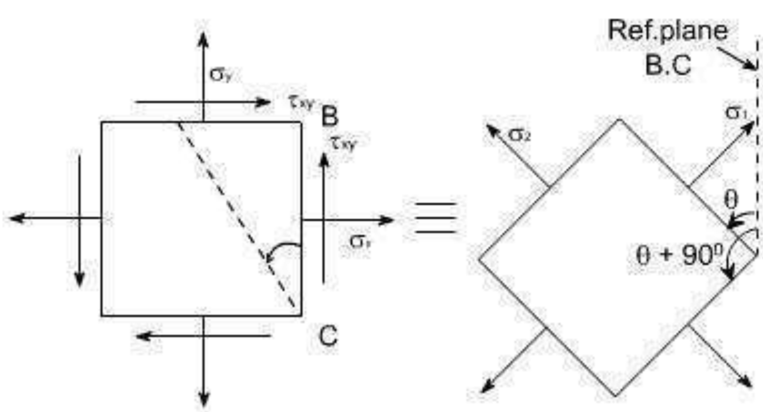
$$= \frac{2 \times 60}{-85 - 25} = +\frac{120}{-110}$$

$$= -\frac{12}{11}$$

$$2\theta = \tan\left(-\frac{12}{11}\right)$$

$$\Rightarrow \theta = -23.74^0$$

Thus, the two principle stresses acting on the two mutually perpendicular planes i.e principle planes may be depicted on the element as shown below:



So this is the direction of one principle plane & the principle stresses acting on this would be s_1 when is acting normal to this plane, now the direction of other principal plane would be 90^0 + q because the principal planes are the two mutually perpendicular plane, hence rotate the another plane q + 90^0 in the same direction to get the another plane, now complete the material element if q is negative that means we are measuring the angles in the opposite direction to the reference plane BC .

